DOCUMENT RESUME

ED 146 219 TM 006 614

AUTHOR Krishnaiah, P. R.
TITLE Simultaneous Inference Procedures for Means.
SPONS AGENCY Air Force Flight Dynamics Lab., Wright-Patterson AFB, Ohio.
PUB DATE [Apr 77]
GRANT AFOSR-77-3239
NOTE 24p.; Paper presented at the American Educational Research Association Symposium on Current Issues in the Analysis of Multivariate Data; Not available in HC due to poor print quality

EDRS PRICE MF-$0.83 Plus Postage. HC Not Available from EDRS.
DESCRIPTORS *Analysis of Variance; *Comparative Analysis; Data Analysis; *Hypothesis Testing; Literature Reviews; Mathematical Models; *Statistical Analysis

ABSTRACT

Some aspects of simultaneous tests for means are reviewed. Specifically, the comparison of univariate or multivariate normal populations based on the values of the means or mean vectors when the variances or covariance matrices are equal is discussed. Tukey's and Dunnett's tests for multiple comparisons of means, Scheffe's method of examining confidence intervals, and the author's finite intersection tests for means are described and compared. The largest characteristic root test and finite intersection tests for multiple comparisons of mean vectors are also described. (Author/MV)

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SIMULTANEOUS INFERENCE PROCEDURES FOR MEANS*

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P. R. KRISHNAIAH

Department of Mathematics and Statistics

University of Pittsburgh

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*Invited Paper presented at the American Educational Research Association Symposium on Current Issues in the Analysis of Multivariate Data. Part of this work was sponsored by the Air Force Flight Dynamics Laboratory, Air Force Systems Command under Grant AFOSR 77-3239.

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1. INTRODUCTION

In many situations, the experimenter is interested in finding out as to which of the populations differ from the other when it is known that the populations are heterogeneous. For example, we have k groups of experimental units and they are subject to k different treatments. Sometimes, we know in advance that there is a significant difference between the effects of these treatments. If we don't know about it, we can always test as to whether or not there is an overall difference between these treatments. Once it is established that there is significant difference between the treatments, the experimenter is interested in finding out as to which of the treatment effects differ from the others. In these situations, the tools of simultaneous tests are very helpful. In this paper, we review some aspects of the simultaneous tests for means. We will restrict our discussion to the comparisons of univariate or multivariate normal populations on the basis of the values of the means or mean vectors when the variances or covariance matrices are equal. For multiple comparison of variances or covariance matrices of normal or multivariate normal populations, the reader is referred to [1, 15, 17, 24, 29, 30, 35, 44, 45].

For various nonparametric simultaneous test procedures, the reader is referred to Miller [9], Gabriel and Sen [14] and Sen and Krishnaiah [56].

2. TUKEY'S AND DUNNETT'S TESTS FOR MULTIPLE COMPARISONS OF MEANS

Consider k univariate normal populations and let the mean and variance of ith population be denoted by μ_i and σ_i^2. We will assume that $\sigma_1^2 = \sigma_2^2 = \ldots = \sigma_k^2 = \sigma^2$. Now, let n_i denote the size of ith sample and x_{ij} be jth observation on ith sample. Also, let $x_{i.} = \sum_{j=1}^{n_i} x_{ij}$, $\bar{x}_{i.} = x_{i.}/n_i$, $s^2 = \sum_{i=1}^{k} \sum_{j=1}^{n_i} (x_{ij}-\bar{x}_{i.})/(n-k)$, $n = \sum_{i=1}^{k} n_i$. We will first discuss Tukey's test for pairwise comparisons of means; his test is restricted to the case when $n_i = m$ for $i = 1,2,\ldots,k$.

Let $H_{ij}: \mu_i = \mu_j$ and $A_{ij}: \mu_i \neq \mu_j$. Using Tukey's test, we accept or reject H_{ij} against A_{ij} according as

$$R_{ij} \lesseqgtr R_\alpha$$

where $P[R<R_\alpha|H] = (1-\alpha)$, $H: \mu_1 = \ldots = \mu_k$, and

$$R_{ij} = \frac{|\bar{x}_{i.} - \bar{x}_{j.}|\sqrt{m}}{s\sqrt{2}}, \quad R = \max_{ij}(R_{ij}).$$

R is known as the studentized Range and the critical values R_α are extensively tabulated by Harter [19].

We will now discuss about the procedures proposed by Dunnett [8] for the comparison of several treatments with a control. Using his procedure, hypotheses $H_{ik}(i=1,2,\ldots,k-1)$, when tested against A_{ik}, are accepted or rejected according as

$$|t_{ik}| \lesseqgtr t_\alpha$$

where

$$P[|t_{ik}| \leq t_\alpha;\ i=1,2,\ldots,k-1|H] = (1-\alpha)$$

and

$$t_{ik} = \frac{(\bar{x}_{i.} - \bar{x}_{k.})}{s\sqrt{\frac{1}{n_i} + \frac{1}{n_k}}}.$$

On the other hand, if we are testing H_{ik} against alternatives of the form $A^*_{ik}: \mu_i > \mu_k$, we accept or reject H_{ik} according as

$$t_{ik} \lesseqgtr t^*_\alpha$$

where

$$P[t_{ik} \leq t^*_\alpha \quad i=1,2,\ldots,k-1|H] = (1-\alpha).$$

When H is true, the joint distribution of $t_{1k},\ldots, t_{k-1,k}$ is a multivariate t distribution with (n-k) degrees of freedom and with Ω as the correlation matrix of the accompanying multivariate normal. When the off-diagonal elements of Ω are equal and Ω is non-signular extensive tables of Krishnaiah and Armitage [38, 39] can be used to get the critical values t_α and t^*_α. We now give the difinition of multivariate t distribution. Let $y_1,\ldots,y_p$ be jointly distributed as a multivariate normal with zero mean vector and covariance matrix $\Sigma = \sigma^2 \Omega$, and z/σ^2 is distributed as a Chi-square with n degrees of freedom and it is distributed independent of $y_1,\ldots,y_p$. Then the joint distribution of $t_1,\ldots,t_p$, where $t_i = \frac{x_i}{\sqrt{z/n}}$ is known to be a central p-variate t distribution with n d.f and with Ω as the correlation matrix of the "accompanying" multivariate normal; here $\Omega = (\rho_{ij})$, $\rho_{ii} = 1$. The multivariate t distribution was first considered by Dunnett and Sobel [10,11] and Cornish [64] independently.

3. SCHEFFE'S METHOD

It is well known that the classical analysis of variance F test is used to test for the equality of means. If this hypotheses is rejected, one can draw inference about the acceptance or rejection of various subhypotheses by Scheffé's method. According to Scheffé's method, we accept or reject any subhypothesis by examining the confidence intervals on the parametric functions which measure departures from the subhypotheses of interest. Suppose, we are interested in testing the subhypotheses $H_1,\ldots,H_q$ simultaneously when H is rejected; here, let H_ℓ: $\lambda_\ell = 0$ where $\lambda_\ell = c_{\ell 1}\mu_1 + \ldots + c_{\ell k}\mu_k$, $c_{\ell 1} + \ldots + c_{\ell k} = 0$, and $c^2_{\ell 1} + \ldots + c^2_{\ell k} = 1$. In our context, Scheffe's procedure is equivalent to the following procedure:

Accept or reject H_i $(i=1,2,\ldots,q)$ against A_i according as

$$F_i \lesseqgtr F_\alpha$$

where

$$P[F_0 \leq (k-1)F_\alpha | H] = (1-\alpha),\ F_0 = \sum_{i=1}^{k} n_i(\bar{x}_i - \bar{x}..)^2/s^2,$$

$$F_\ell = \frac{(\hat{\lambda}_\ell)^2}{s^2 d_\ell},\quad d_\ell = \sum_{j=1}^{k} c_{\ell j}^2/n_j\ ,$$

$$\hat{\lambda}_\ell = c_{\ell 1}\bar{x}_1. + \ldots + c_{\ell k}\bar{x}_k,\ \Lambda_\ell : \lambda_\ell \neq 0, \lambda_\ell = \sum_{j=1}^{k} c_{\ell j}\mu_j.$$

Here F_α is the upper α% value of F variate with (k-1,n-k) d.f.

When σ^2 is known, the analogous method is as follows:

Accept or reject H_i according as

$$\chi_\ell^2 \lesseqgtr \chi_\alpha^2$$

where

$$P[\chi_0^2 \leq \chi_\alpha^2(k-1) | H] = (1-\alpha), \chi_\ell^2 = \frac{(\hat{\lambda}_\ell)^2}{d_\ell \sigma^2}, \chi_0^2 = \sum_{i=1}^{k} n_i(\bar{x}_i. - \bar{x}..)^2/\sigma^2$$

and χ_α^2 is upper α% value of chi-square variate with (k-1) degrees of freedom.

4. FINITE INTERSECTION TESTS FOR MEANS

We will consider special cases of the Finite Intersection Tests proposed by Krishnaiah [26, 28, 33] for one-sided and two-sided alternatives. First, we will discuss these tests against two-sided alternatives. In other words, we wish to test $H_1,\ldots,H_q$ against $A_1,\ldots, A_q$ respectively. If we use the Finite Intersection Tests, we accept or reject H_i according as

$$F_i \lesseqgtr F_{\alpha 1}$$

where

$$P[F_i \leq F_{\alpha 1}\ ;\ i=1,2,\ldots,q | H] = (1-\alpha),$$

$$F_i = t_i^2 \text{ and } t_i = \frac{\hat{\lambda}_i}{(d_i s^2)^{1/2}}.$$

When H is true, the joint distribution of $F_1,\ldots, F_q$ is a multivariate F distribution with (1,n-k) degrees of freedom and with Ω as the correlation matrix of the "accompanying" multivariate normal; the distribution is singular or non-singular according as Ω is singular or non-singular. The multivariate F distribution encountered here was introduced by Krishnaiah (1965). Some tables for this distribution are given in Schuurmarn, Krishnaiah and Chattopadhyay [55]. Tukey's test for simultaneous pairwise comparisons of means and Dunnett's test for comparisons of various population means with that of the control are special cases of the above test procedure. The only difference is that we are working with t_i^2 as test statistics whereas they worked with $|t_i|$ as test statistics.

The simultaneous confidence intervals associated with the procedure discussed above are derived as follows. We know that the joint distribution of $\hat{\lambda}_1,\ldots,\hat{\lambda}_q$ under the null hypothesis H is the same as the joint distribution of $(\hat{\lambda}_1-\lambda_1),\ldots(\hat{\lambda}_q-\lambda_q)$ under $\bigcap_{i=1}^{q} A_i$, that is, the joint distribution is a multivariate normal with zero mean vector and with certain covariance matrix.

So,

$$P[F_i \leq F_{\alpha 1};\ i=1,2,\ldots,q|H] = P[F_i^* \leq F_{\alpha 1};\ i=1,2,\ldots,q|\bigcap_{i=1}^{q} A_i]$$

$$=(1-\alpha)$$

$$\text{But } F_i^* \leq F_{\alpha 1} \quad |\hat{\lambda}_i-\lambda_i| \leq \sqrt{F_{\alpha 1} d_\ell s^2}.$$

So, we get the following simultaneous confidence intervals:

$$P[\hat{\lambda}_\ell - \sqrt{F_{\alpha 1} d_\ell s^2} \leq \lambda_\ell \leq \hat{\lambda}_\ell + \sqrt{F_{\alpha 1} d_\ell s^2};\ \ell=1,2,\ldots,q]=(1-\alpha).$$

Before discussing about the evaluation of the critical values $F_{\alpha l}$, we will consider the problem of testing $H_1,\ldots,H_q$ simultaneously against $A_1^*,\ldots,A_q^*$ where A_ℓ^*: $\lambda_\ell > 0$. If we use the Finite Intersection Tests, we accept or reject H_l according as

$$t_\ell \overset{\leq}{} t_{\alpha l}$$

where

$$P[t_\ell \leq t_{\alpha l};\ \ell=1,2,\ldots,q|H] = (1-\alpha) .$$

As before, we can derive the following simultaneous confidence intervals associated with the above procedure:

$$P[\hat{\lambda}_\ell - t_{\alpha l}\sqrt{s^2 d_\ell} \leq \lambda_\ell\ ;\ \ell=1,2,\ldots,q] = (1-\alpha).$$

Sometimes, the experimenter may be interested in testing the hypotheses $H_1,\ldots,H_q$ simultaneously against the respective alternatives $A_1^{**},\ldots,A_q^{**}$ where A_ℓ^{**}: $\lambda_\ell < 0$. In this case, using the Finite Intersection Tests, we accept or reject H_ℓ according as

$$t_\ell \overset{\geq}{} t_{\alpha 2}$$

where

$$P[t_\ell \geq t_{\alpha 2};\ \ell=1,2,\ldots,q|H]=(1-\alpha).$$

The simultaneous confidence intervals associated with the above procedure are given by

$$P[\hat{\lambda}_\ell - t_{\alpha 2}\sqrt{s^2 d_\ell} \geq \lambda_\ell\ ;\ \ell=1,2,\ldots,q]=(1-\alpha).$$

We will now discuss about the evaluation of the critical values $F_{\alpha l}$, $t_{\alpha l}$ and $t_{\alpha 2}$.

When H is true the joint distribution of $F_1,\ldots,F_q$ is a q-variate F distribution with (1,n-k) degrees of freedom and with Ω as the correlation matrix of the "accompanying" multivariate normal where $\Omega = (\rho_{i,j})$ and

$$\rho_{ij} = \frac{1}{\sqrt{d_i d_j}} \sum_{r=1}^{k} \frac{c_{ir}\, c_{jr}}{n_r}$$

(For example, when $n_1 = \ldots = n_k = m$, H_i: $\mu_i - \mu_k = 0$, H_j: $\mu_j - \mu_k = 0$, then $\rho_{ij} = \frac{1}{2}$). When Ω is non-singular and $\rho_{ij} = \rho, (i \neq j = 1,2,\ldots,k)$, the values of $F_{\alpha 1}$ can be obtained from the tables of Krishnaiah and Armitage [39]. In other situations, we can get bounds on F_α as described below: According to Poincare's formula, it is known that

$$P(E_1 E_2 \ldots E_q) = 1 - \sum_{i=1}^{q} P(E_i^c) + \sum_{i}\sum_{j} P(E_i^c E_j^c) + \ldots$$

$$+(-1)^q P(E_1^c \ldots E_q^c)$$

where E_i^c is the complement of the event E_i. Using the above, we can construct monotonic upper and lower bounds as follows (e.g., see Feller [12;p. 100]):

Let $S_r = 1 - \sum_{i=1}^{q} P(E_i^c) + \ldots + (-1)^r \sum_{i_1 < \ldots < i_r} P(E_{i_1}^c \ldots E_{i_r}^c)$

Then $S_1 \geq S_3 \geq S_5 \ldots \geq S_q \geq S_2 \geq S_4 \geq \ldots$.

In our context, we will use

$$1 - \sum_{i=1}^{q} P(F_i \geq F_{\alpha 1} \mid H_i) \leq P[F_i \leq F_{\alpha 1};\ i = 1,2,\ldots,q \mid H]$$

$$\leq 1 - \sum_{i=1}^{q} P(F_i \geq F_{\alpha 1} \mid H_i) + \sum_{i<j} P(F_i \geq F_{\alpha 1}, F_j \geq F_{\alpha 1} \mid H_i \cap H_j).$$

But $P(F_i \geq F_{\alpha 1}, F_j \geq F_{\alpha 2} \mid H_i \cap H_j) = 1 - P(F_i \leq F_{\alpha 1} \mid H_i) - P(F_j \leq F_{\alpha 1} \mid H_j)$

$$+P(F_i \leq F_{\alpha 1}, F_j \leq F_{\alpha 1} \mid H_i \cap H_j).$$

Also, when H_i is true, F_i is distributed as a central F with (1,n-k) df. So values of terms of the form $P(F_i \leq F_{\alpha 1} \mid H_i)$ can be easily obtained from tables. Values of terms of the form $P(F_i \leq F_{\alpha 1}, F_j \leq F_{\alpha 1} \mid H_i \cap H_j)$ can be obtained by using the extensive tables of Krishnaiah, Armitage and Breiter [41]. Now, let $F_{\alpha 1L}$ and F_{L1U} be the lower and upper bounds of $F_{\alpha 1}$. Then, we accept or reject H_i according as $F_i \leq F_{\alpha 1L}$ or $F_i \geq F_{\alpha 1U}$. If F_i lies between $F_{\alpha 1L}$ and $F_{\alpha 1U}$, we will not make any decision. We can similarly compute upper and lower bounds on $t_{\alpha 1}$ and $t_{\alpha 2}$ and draw inference in a similar way for one-sided tests when Ω is singular and/or $\rho_{ij} \neq \rho$ for $i \neq j = 1,2,\ldots, q$. Tables of student's t distribution and extensive tables of Krishnaiah, Armitage and Breiter [40] for the probability integral of the bivariate t can be used in constructing these bounds. When Ω is non-singular and $\rho_{ij} = \rho$ for $i \neq j = 1, 2,\ldots,q$, tables of Krishnaiah and Armitage [38] can be used to obtain exact values for $t_{\alpha 1}$.

When σ^2 is known, the tests discussed in this section can be easily modified.

5. COMPARISON OF SCHEFFÉ'S METHOD WITH FINITE INTERSECTION TESTS

When the total hypothesis H is rejected, Scheffé's method gives a procedure to draw inference on all possible constrasts. But the experimenter is interested in drawing inference on a finite number of contrasts only and Scheffe's method is less sensitive than the Finite Intersection Tests which are designed to test only the contrasts of interest. We will now elaborate this point.

The critical value in classical F test is chosen such that

$$P[F_0 \leq F_\alpha(k-1) \mid H] = (1-\alpha$$

when $F_0 = s_H^2/s^2$ and $s_H^2 = \sum_{i=1}^{k} n_i(\bar{x}_{i.} - \bar{x}_{..})^2$ The critical value of the Finite Intersection Test for two-sided alternatives is chosen such that

$$P[\max (F_1,\ldots,F_q) \leq F_{\alpha l} \mid H] = (1-\alpha).$$

But $\max (F_1,\ldots,F_q) \leq (k-1)F_0$ and hence $F_{\alpha l} \leq (k-1)\, F_\alpha$. If we use Scheffé's method, the power of the test for testing H_i against A_i is given by P_{1i} where

$$P_{1i} = P\{F_i \geq (k-1)F_\alpha \mid A_i\}.$$

Similarly, if we use the Finite Intersection Tests, the power of the test for testing, H_i against A_i is given by

$$P_{2i} = P\{F_i \geq F_{\alpha l} \mid A_i\}.$$

Since $F_{\alpha l} \leq (k-1)F_\alpha$, $P_{2i} \geq P_{1i}$. Also, since the confidence intervals are obtained by inversion of the power functions, Finite Intersection Tests yield shorter (in Neyman sense) confidence intervals than Scheffé's method; here we note that shortness in Neyman sense is defined in terms of the probability of covering wrong values.

6. LARGEST CHARACTERISTIC ROOT TEST

We will now discuss various multivariate generalizations of the tests discussed in sections 2-4. We will now discuss about multiple comparison tests based on the largest characteristic root test. To discuss this procedure, let x_{itu} denote t th observation on uth variate in ith sample and let $\underset{\sim}{x}'_{it} = (x_{it1},\ldots,x_{itp})$. Then the observations in each vector are distributed jointly as a multivariate normal with covariance matrix Σ and means given by

$$E(\underset{\sim}{x}'_{jt}) = \underset{\sim}{\mu}'_i = (\mu_{i1},\ldots,\mu_{ip})$$

Now, let

$$\underset{\sim}{\lambda}_\ell = c_{\ell 1}\underset{\sim}{\mu}_1 + \ldots + c_{\ell k}\ \underset{\sim}{\mu}_k$$

$$\hat{\lambda}_\ell = c_{\ell 1}\ \hat{\underset{\sim}{\mu}}_1 + \ldots + c_{\ell k}\ \hat{\underset{\sim}{\mu}}_k$$

where $\hat{\underset{\sim}{\mu}}_i = (\bar{x}_{i.1},\ldots,\bar{x}_{i.p})$ and $\bar{x}_{i.j}$ is ith sample mean on jth variate. In addition, let $S_H = (s^0_{ab})$ where $s^0_{ab} = \sum_{i=1}^{k} n_i(\bar{x}_{i.a} - \bar{x}_{..a})(\bar{x}_{i.b} - \bar{x}_{..b})$

$$\bar{x}_{..a} = \frac{1}{n} \sum_i \sum_j x_{ija}.$$

Also, $S = (s_{ab})$ where

$$s_{ab} = \frac{1}{(n-k)} \sum_{i=1}^{k} \sum_{j=1}^{n_i} (x_{ija} - \bar{x}_{i.a})(x_{ijb} - \bar{x}_{i.b}).$$

S. N. Roy proposed a procedure for testing the hypothesis H where H: $\underset{\sim}{\mu}_1 = \ldots = \underset{\sim}{\mu}_k$. According to this procedure, we accept or reject H according as

$$c_L(S_H S^{-1}) \lesseqgtr c_\alpha(k-1)$$

where c_α is chosen such that

$$P[c_L(S_H S^{-1}) \leq c_\alpha(k-1) \mid H] = (1-\alpha)$$

and $c_L(A)$ denotes the largest characteristic root of A. The critical values c_α can be obtained from the tables of Heck [21] and Pillai [48]. This test procedure is known as the "largest characteristic root test". In the univariate case, this reduces to the classical ANOVA F test. When the overall hypothesis H is rejected, its subhypotheses can be tested by examining the confidence intervals associated with the test; these confidence intervals are derived by Roy and Bose [51]. In out context, this procedure is equivalent to accepting or rejecting H_i according as

$$T_i^2 \lesseqgtr c_\alpha(k-1)$$

where

$$P[c_L(S_H S^{-1}) \leq c_\alpha \ \ (k-1)|H] = (1-\alpha),$$

$$H_i: c_{i1}\mu_1 + \ldots + c_{ik}\mu_k = 0$$

$$T_i^2 = \hat{\lambda}_i' \ S^{-1} \ \hat{\lambda}_i/d_i.$$

When Σ is known, we accept or reject H_i according as

$$\chi_i^2 \lesseqgtr \chi_\alpha^2 \ (k-1)$$

where

$$P[c_L(S_H \Sigma^{-1}) \leq (k-1)\chi_\alpha^2|H] = (1-\alpha), \ \chi_i^2 = \hat{\lambda}_i' \ \Sigma^{-1}\hat{\lambda}_i/d_i \ .$$

7. T^2_{MAX} AND χ^2_{MAX} TESTS

We now consider the multiple comparisons of mean vectors by using Hotelling's T^2 or χ^2 statistics according as Σ is unknown or known. Since $H_\ell = \bigcap_{a\neq 0} H_{\ell a}$ where $H_{\ell a}: \lambda_\ell' a = 0$, we can test $H_1,\ldots,H_q$ simultaneously by testing $H_{\ell a}$'s simultaneously and using Roy's union-intersection as follows:

We accept H_ℓ if

$$F_{\ell a} \leq T_\alpha^2 \text{ for all non-null } a$$

and reject H_ℓ otherwise where T_α^2 is chosen such that

$$P[F_{\ell a} \leq T_\alpha^2, \ \ell = 1,2,\ldots,q \text{ and non-null } a|H] = (1-\alpha)$$

$$\text{and } F_{\ell a} = \frac{(\hat{\lambda}_\ell' \ a)^2}{a' S a \ d_\ell}.$$

But $\bigcap_{a\neq 0} [F_{\ell a} \leq T_\alpha^2] \Leftrightarrow T_\ell^2 \leq T_\alpha^2$. So, we accept or reject H_ℓ according as

$$T_\ell^2 \lesseqgtr T_\alpha^2$$

where T_α^2 is chosen such that

$$P[T_\ell^2 \leq T_\alpha^2 \; ; \; \ell = 1,2,\ldots, \; q|H] = (1-\alpha).$$

The exact joint distribution of the T_ℓ^2 statistics is not known. Based on Poincare's formula, we can use the following inequalities:

$$1-P_0 \leq P \leq 1-P_0+P_1$$

where

$$P_0 = \sum_{\ell=1}^{q} P[T_\ell^2 \geq T_\alpha^2|H]$$

$$P_1 = \sum_{\ell<\ell'} P[T_\ell^2 \geq T_\alpha^2, \; T_{\ell'}^2 \geq T_\alpha^2|H].$$

When H is true, T_ℓ^2 is distributed as F with $(p,n-k-p+1)$ d.f. and hence P_0 can be evaluated. When the sample size is at least moderately large, Siotani suggested an approximation for the evaluation of P_1; he gave some tables for some cases. The test based on the largest T^2 was formulated by Roy and Bose [51] for simultaneous pairwise comparisons of mean vectors. Now, let us discuss the problem of testing H_ℓ's when Σ is known. In this case, we use the following procedure.

Accept or reject H_ℓ according as

$$\chi_\ell^2 \lesseqgtr \chi_\alpha^2$$

where

$$P[\chi_\ell^2 \leq \chi_\alpha^2 \; ; \; \ell = 1,2,\ldots,q|H] = (1-\alpha)$$

and

$$\chi_\ell^2 = \frac{(\hat{\lambda}_\ell' \, \Sigma^{-1} \hat{\lambda}_\ell)}{d_\ell} \; .$$

When H is true, the joint distribution of these test statistics is a multivariate chi-square distribution with p d.f.

The simultaneous confidence intervals associated with T^2_{Max} and χ^2_{Max} tests can be obtained without difficulty. For some special cases, the confidence bounds are given by Siotani [59].

8. FINITE INTERSECTION TESTS FOR MULTIPLE COMPARISONS OF MEAN VECTORS

In several situations, the experimenter is interested in testing the hypotheses of the form H_{ℓ,a_j} simultaneously against alternatives of the form A_{ℓ,a_j} where H_{ℓ,a_j}: $\lambda_\ell' a_j = 0$ and A_{ℓ,a_j}: $\lambda_\ell' a_j \neq 0$ and a_j's are known. If, in particular, a_j is such that its jth element is 1 and the other elements are zero, H_{ℓ,a_j}: $c_{\ell 1}\mu_{1j}+\ldots+c_{\ell k}\mu_{kj}=0$, where μ_{ij} is ith population mean on jth variate as defined before. The elements of a_j can be looked upon as weights assigned to different variates. We can test $H_{\ell a_j}$'s as follows:

Accept or reject $H_{\ell a_j}$ according as

$$F_{\ell a_j} \lesseqgtr F_\alpha$$

where F_α is chosen such that

$$P[F_{\ell a_j} \leq F_\alpha \;;\; \ell = 1,2,\ldots,q \;,\; j=1,2,\ldots,r \mid H] = (1-\alpha),$$

$$H = \bigcap_{j=1}^{r}\bigcap_{\ell=1}^{q} H_{\ell a_j} \text{ and } F_{\ell a_j} = \frac{(\lambda_\ell' a_j)^2}{a_j' S a_j \; d_\ell}$$

The joint distribution of numerators is a multivariate chi-square and the joint distribution of the denominators is another multivariate chi-square; the joint distribution of $F_{\ell a_j}$'s involve nuisance parameters. So, we can obtain the bounds on the critical values by using Bonferroni's inequality.

Similarly, we can test $H_{\ell,\underset{\sim}{a}_j}$'s against one-sided alternative. We will now discuss about testing $H_{\ell\underset{\sim}{a}_j}$'s against $A_{\ell\underset{\sim}{a}_j}$'s when Σ is known.

Accept or reject $H_{\ell\underset{\sim}{a}_j}$ according as

$$\chi^2_{\ell\underset{\sim}{a}_j} \leq \chi^2_\alpha$$

where

$$P[\chi^2_{\ell\underset{\sim}{a}_j} \leq \chi^2_\alpha;\ \ell = 1,2,\ldots,q\ ;\ j = 1,2,\ldots,r|H] = (1-\alpha)$$

$$\chi^2_{\ell\underset{\sim}{a}_j} = \frac{(\hat{\underset{\sim}{\lambda}}'_\ell\ \underset{\sim}{a}_j)^2}{d_\ell\ \underset{\sim}{a}'_j\ \Sigma\underset{\sim}{a}_j}$$

The joint distribution of $\chi^2_{\ell\underset{\sim}{a}_j}$'s is a multivariate χ^2 with one degree of freedom and so the exact critical value (or lower and upper bounds on them) can be computed. Similarly, we can consider tests against one-sided alternatives when Σ is known.

We will now consider alternative Finite Intersection Test procedures which are based on conditional distributions. For various details of these procedures the reader is referred to Krishnaiah [26,33].

Now, let $\underset{\sim}{y}'_{i,j+1} = (x_{i1j+1},\ldots,x_{in_ij+1})$, $Y_{ij} = [\underset{\sim}{y}_{i1},\ldots,\underset{\sim}{y}_{ij}]$ and $\underset{\sim}{\theta}'_{ij} = (\mu_{i1},\ldots,\mu_{ij})$. We know that, when Y_{ij} is fixed, the distribution of the elements of $\underset{\sim}{y}'_{i,j+1}$ are independently and normally distributed with variance σ^2_{j+1} and means given by

$$(8.1) \qquad E_c(x_{it,j+1}) = \eta_{i,j+1} + \underset{\sim}{\beta}'_j \begin{pmatrix} x_{it1} \\ \cdot \\ \cdot \\ \cdot \\ x_{itj} \end{pmatrix} \qquad t = 1,2,\ldots,n_i$$

where

$$\eta_{i,j+1} = \mu_{i,j+1} - \underset{\sim}{\beta}_j' \underset{\sim}{\theta}_{ij}, \quad \sigma^2_{j+1} = \frac{|\Sigma_{j+1}|}{|\Sigma_j|}$$

$$\underset{\sim}{\beta}_j = \Sigma_j^{-1} \begin{pmatrix} \sigma_{1,j+1} \\ \vdots \\ \sigma_{j,j+1} \end{pmatrix}, \quad \beta_0 = 0,$$

$\Sigma = (\sigma_{rs})$ and Σ_j is the top jxj left hand corner of Σ. Now, let

$$(n-k)\, s_j^2 = \frac{|S_{j+1}|}{|S_j|}, \quad t_{\ell j} = \frac{(c_{\ell 1}\hat{\eta}_{1j} + \ldots + c_{\ell k}\hat{\eta}_{kj})\sqrt{(n-k-j+1)}}{\sqrt{s_j^2\, d_\ell}},$$

$$\text{Var}(c_{\ell 1}\hat{\eta}_{1j} + \ldots + c_{\ell k}\hat{\eta}_{kj}) = d_\ell\, \sigma^2_{j+1}, \quad F_{\ell j} = t^2_{\ell j}$$

$$\hat{\eta}_{ij} = \hat{\mu}_{ij} - \hat{\underset{\sim}{\beta}}_j' \hat{\underset{\sim}{\theta}}_{ij}, \quad \hat{\mu}_{ij} = \bar{x}_{ij}, \quad \hat{\underset{\sim}{\theta}}_{ij}' = (\hat{\mu}_{i1}, \ldots, \hat{\mu}_{ij})$$

$$\hat{\underset{\sim}{\beta}}_j' = (s_{1,j+1}, \ldots, s_{j,j+1})\, S_j^{-1}, \quad \hat{\beta} = 0,$$

and S_j is the top left-hand corner of the within group mean squares and cross products matrix S. Also, let

$$H_{\ell j} : c_{\ell 1}\eta_{1j} + \ldots + c_{\ell k}\eta_{kj} = 0.$$

Then $H_\ell = \bigcap_{j=1}^{p} H_{\ell j}$. So, the problem of testing $H_1, \ldots, H_q$ simultaneously is equivalent to testing $H_{\ell j}$'s simultaneously. So, we propose the following procedure for testing $H_1, \ldots, H_q$ simultaneously against two-sided alternatives.

Accept H_ℓ if and only if

$$F_{\ell j} \leq f_{j\alpha} \quad \text{for } j=1,2,\ldots,p$$

where the constants $f_{j\alpha}$ are chosen such that

$$P[F_{\ell j} \leq f_{j\alpha} \; ; \; j=1,2,\ldots,p \; \ell=1,2,\ldots,q|H]$$

$$= \prod_{j=1}^{p} P[F_{\ell j} < f_{j\alpha} \; ; \; \ell= 1,2,\ldots,q|H]= (1-\alpha).$$

For given j, the joint distribution of $F_{1j},\ldots,F_{qj}$ is a multivariate F distribution with (1,n-k-j+1) d.f. In the univariate case, this test reduces to a test discussed in section 4. We will now discuss about procedures for testing $H_1,\ldots, H_q$ simultaneously against the alternatives $A_1,\ldots, A_q$ where A_ℓ: $c_{\ell 1}\mu_1+\ldots + c_{\ell k}\mu_k > 0$. The test procedure in this case is to accept H_ℓ if and only if

$$t_{\ell j} < t_{j\alpha} \qquad j=1,2,\ldots,p$$

where $t_{j\alpha}$'s are chosen such that

$$P[t_{\ell j} \leq t_{j\alpha} \; ; \; j=1,2,\ldots,p \quad \ell= 1,2,\ldots,q|H]$$

$$= \prod_{j=1}^{p} P[t_{\ell j} \leq t_{j\alpha}; \quad \ell= 1,2,\ldots, q|H] = (1-\alpha)$$

For given j, the joint distribution of $t_{1j},\ldots,t_{qj}$ is a multivariate t distribution with (1,n-k-j+1) d.f.

J. Roy [49] proposed a test procedure for testing the equality of mean vector by using conditional distributions. We can test the subhypotheses of interest against two-sided alternatives by using the confidence intervals associated with his procedure. But the power of the tests proposed by Krishnaiah is larger than J. Roy's test when one is testing these subhypotheses.

9. GENERAL REMARKS

For a detailed discussion of various simultaneous test procedures on mean vectors under growth curve models, the reader is referred to Krishnaiah [31,33].

The problem of testing the hypotheses on covariance structures have been considered by several authors in the literature. The author is now investigating simultaneous tests on mean vectors when the covariance matrices of the multivariate normal populations have general structures of the form considered by Krishnaiah and Lee [42].

Computer programs for the application of the procedures discussed in the present paper will be ready sometime this summer.

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